APPENDIX A
DERIVATION OF THE SOLUTION IN (15)

Since the cost in (15) is quadratic, *i.e.*, convex, we differentiate it with respect to $\boldsymbol{\mathcal{X}}$ and set it to $\mathbf{0}$, *i.e.*,

$$\begin{aligned}
&\frac{\partial}{\partial \boldsymbol{\mathcal{X}}}\left(\frac{1}{2}\left\|\sqrt{\mathcal{F}_{\mathrm{att}}(\boldsymbol{\mathcal{D}})}\odot(\boldsymbol{\mathcal{L}}_k * \boldsymbol{\mathcal{R}}_k - \boldsymbol{\mathcal{X}})\right\|_F^2 \right.\\
&\qquad\qquad\left. + \frac{\mu_k+\alpha_k}{2}\left\|\boldsymbol{\mathcal{X}} - \boldsymbol{\Psi}_{\boldsymbol{\mathcal{X}},k}\right\|_F^2\right)\\
&= -\sqrt{\mathcal{F}_{\mathrm{att}}(\boldsymbol{\mathcal{D}})}\odot\left(\sqrt{\mathcal{F}_{\mathrm{att}}(\boldsymbol{\mathcal{D}})}\odot(\boldsymbol{\mathcal{L}}_k * \boldsymbol{\mathcal{R}}_k - \boldsymbol{\mathcal{X}})\right)\\
&\qquad\qquad + (\mu_k+\alpha_k)(\boldsymbol{\mathcal{X}} - \boldsymbol{\Psi}_{\boldsymbol{\mathcal{X}},k}) = \mathbf{0} \qquad (34)\\
&\Leftrightarrow\ (\mathcal{F}_{\mathrm{att}}(\boldsymbol{\mathcal{D}}) + \mu_k\mathbf{1} + \alpha_k\mathbf{1})\odot\boldsymbol{\mathcal{X}}\\
&\qquad = \mathcal{F}_{\mathrm{att}}(\boldsymbol{\mathcal{D}})\odot(\boldsymbol{\mathcal{L}}_k * \boldsymbol{\mathcal{R}}_k) + (\mu_k+\alpha_k)\boldsymbol{\Psi}_{\boldsymbol{\mathcal{X}},k}. \qquad (35)
\end{aligned}$$

Since each element of $\boldsymbol{\mathcal{X}}$ on the left-hand side of (35) is multiplied by a scalar value $\big([\mathcal{F}_{\mathrm{att}}(\boldsymbol{\mathcal{D}})](i_1,i_2,i_3)+\mu_k+\alpha_k\big)$, we obtain the closed-form solution in (15) by performing element-wise division.

APPENDIX B
DERIVATION OF THE SOLUTION IN (21)

The optimization problem in (21) can be reformulated as

$$\begin{aligned}
&\underset{\boldsymbol{\mathcal{S}}}{\text{minimize}} && \|\boldsymbol{\mathcal{S}} - \boldsymbol{\Psi}_{\boldsymbol{\mathcal{S}},k}\|_F^2\\
&\text{subject to} && \|\mathcal{P}_{\boldsymbol{\Omega}}(\boldsymbol{\mathcal{S}})\|_F \le \delta_k.
\end{aligned} \qquad (36)$$

We solve this problem by using Theorem 1 from [1]. For the elements $(i_1,i_2,i_3)\notin\boldsymbol{\Omega}$, the solution is given by

$$\mathcal{P}_{\boldsymbol{\Omega}^{\mathrm{c}}}(\boldsymbol{\mathcal{S}}_{k+1}) = \mathcal{P}_{\boldsymbol{\Omega}^{\mathrm{c}}}(\boldsymbol{\Psi}_{\boldsymbol{\mathcal{S}},k}), \qquad (37)$$

since it is apparent that $\boldsymbol{\mathcal{S}}_{k+1}(i_1,i_2,i_3) = \boldsymbol{\Psi}_{\boldsymbol{\mathcal{S}},k}(i_1,i_2,i_3)$. For the elements $(i_1,i_2,i_3)\in\boldsymbol{\Omega}$, we first denote $\boldsymbol{\mathcal{Z}} = \mathcal{P}_{\boldsymbol{\Omega}}(\boldsymbol{\mathcal{S}})$ and $\boldsymbol{\mathcal{W}} = \mathcal{P}_{\boldsymbol{\Omega}}(\boldsymbol{\Psi}_{\boldsymbol{\mathcal{S}},k})$ for simpler notations. Then, the optimization in (36) can be rewritten as

$$\begin{aligned}
&\underset{\boldsymbol{\mathcal{Z}}}{\text{minimize}} && \|\boldsymbol{\mathcal{Z}} - \boldsymbol{\mathcal{W}}\|_F^2\\
&\text{subject to} && \|\boldsymbol{\mathcal{Z}}\|_F^2 \le \delta_k^2.
\end{aligned} \qquad (38)$$

We define the Lagrangian function for (38) as

$$\mathcal{L}(\boldsymbol{\mathcal{Z}},\nu) = \|\boldsymbol{\mathcal{Z}} - \boldsymbol{\mathcal{W}}\|_F^2 + \nu(\|\boldsymbol{\mathcal{Z}}\|_F^2 - \delta_k^2),$$

where ν is the Lagrange multiplier for the constraint. Then, the optimal $\boldsymbol{\mathcal{Z}}$ can be obtained by solving the Karush-Kuhn-Tucker conditions [2], *i.e.*,

$$\|\boldsymbol{\mathcal{Z}}\|_F - \delta_k \le 0, \qquad (39)$$

$$\nu \ge 0, \qquad (40)$$

$$\nu(\|\boldsymbol{\mathcal{Z}}\|_F - \delta_k) = 0, \qquad (41)$$

$$(1+\nu)\boldsymbol{\mathcal{Z}} - \boldsymbol{\mathcal{W}} = \mathbf{0}, \qquad (42)$$

where $\mathbf{0}$ is the zero tensor. By substituting $\boldsymbol{\mathcal{Z}}$ from (42) into (41), we obtain

$$\nu(\|\boldsymbol{\mathcal{Z}}\|_F - \delta_k) = \frac{\nu}{1+\nu}\left\{\|\boldsymbol{\mathcal{W}}\|_F - (1+\nu)\delta_k\right\} = 0. \qquad (43)$$

We consider two cases.

Case 1: $\|\boldsymbol{\mathcal{W}}\|_F < \delta_k$. Then, $\|\boldsymbol{\mathcal{W}}\|_F - (1+\nu)\delta_k < 0$, and thus $\nu = 0$ from (40). Therefore, we have $\boldsymbol{\mathcal{Z}} = \boldsymbol{\mathcal{W}}$.

Case 2: $\|\boldsymbol{\mathcal{W}}\|_F \ge \delta_k$. In this case, $\|\boldsymbol{\mathcal{W}}\|_F - (1+\nu)\delta_k = 0$. Then, from (42), we have $\boldsymbol{\mathcal{Z}} = \frac{\delta_k}{\|\boldsymbol{\mathcal{W}}\|_F}\boldsymbol{\mathcal{W}}$.

By combining the two cases, we obtain the optimal $\boldsymbol{\mathcal{Z}}$ as

$$\boldsymbol{\mathcal{Z}} = \mathcal{P}_{\boldsymbol{\Omega}}(\boldsymbol{\mathcal{S}}_{k+1}) = \min\left\{\frac{\delta_k}{\|\mathcal{P}_{\boldsymbol{\Omega}}(\boldsymbol{\Psi}_{\boldsymbol{\mathcal{S}},k})\|_F}, 1\right\}\mathcal{P}_{\boldsymbol{\Omega}}(\boldsymbol{\Psi}_{\boldsymbol{\mathcal{S}},k}). \qquad (44)$$

Since $\boldsymbol{\mathcal{S}}_{k+1} = \mathcal{P}_{\boldsymbol{\Omega}^{\mathrm{c}}}(\boldsymbol{\mathcal{S}}_{k+1}) + \mathcal{P}_{\boldsymbol{\Omega}}(\boldsymbol{\mathcal{S}}_{k+1})$, from (37) and (44), we obtain the solution in (21).

REFERENCES

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